Syntactically Characterizing Local-to-Global Consistency in ORD-Horn

Michał Wrona

Linköping University
SE-581 83 Linköping, Sweden
michal.wrona@liu.se

CP 2012, Québec City

Outline

Outline

Outline

1. Constraint Satisfaction Problems
2. Local-to-Global Consistency
3. ORD-Horn Constraint Languages

Constraint Satisfaction Problem(Γ)

Let Γ be a relational τ-structure over some domain D.

CSP(Γ) — for finite τ

Instance: A primitive positive formula ϕ of the form $R_1(x^1_1, \ldots, x^1_{n_1}) \wedge \ldots \wedge R_k(x^k_1, \ldots, x^k_{n_k})$, where $R_1, \ldots, R_k \in \tau$.
Question: Is ϕ satisfiable in Γ?

Constraint Satisfaction Problem(Γ)

Let Γ be a relational τ-structure over some domain D.

CSP(Γ) — for finite τ

Instance: A primitive positive formula ϕ of the form $R_1(x_1^1, \ldots, x_{n_1}^1) \wedge \ldots \wedge R_k(x_1^k, \ldots, x_{n_k}^k)$, where $R_1, \ldots, R_k \in \tau$.
Question: Is ϕ satisfiable in Γ?

Many problems may be modelled as CSP(Γ):

- boolean satisfiability problems, e.g., k-SAT;
- graph homomorphism problems, e.g., k-colouring;
- network satisfaction problems in spatial-temporal reasoning, for *qualitative calculi* such as *point algebra*, *Allen's interval algebra*, *region connection calculi*.

Constraint Satisfaction Problem(Γ)

Let Γ be a relational τ-structure over some domain D.

CSP(Γ) — for finite τ

Instance: A primitive positive formula ϕ of the form
$R_1(x_1^1, \ldots, x_{n_1}^1) \wedge \ldots \wedge R_k(x_1^k, \ldots, x_{n_k}^k)$, where $R_1, \ldots, R_k \in \tau$.
Question: Is ϕ satisfiable in Γ?

Many problems may be modelled as CSP(Γ):

- boolean satisfiability problems, e.g., k-SAT;
- graph homomorphism problems, e.g., k-colouring;
- network satisfaction problems in spatial-temporal reasoning, for *qualitative calculi* such as *point algebra*, *Allen's interval algebra*, *region connection calculi*.

All structures considered here have a first-order definition in $(\mathbb{Q}; <)$, they are all ω-categorical.

A countably infinite structure Γ is ω-categorical if all countable models of the first-order theory of Γ are isomorphic.

Network Satisfaction Problem for Point Algebra

A point algebra $\mathcal{P}$ is a set of all binary relations $\mathcal{R}_2(\mathbb{Q})$ with a first order definition over $(\mathbb{Q}; <)$:

- $<, >, \neq, \leq, \geq, =, \mathbb{Q}^2, \emptyset$.

Network Satisfaction Problem for Point Algebra

A point algebra $\mathcal{P}$ is a set of all binary relations $\mathcal{R}_2(\mathbb{Q})$ with a first order definition over $(\mathbb{Q}; <)$:

- $<, >, \neq, \leq, \geq, =, \mathbb{Q}^2, \emptyset$.

The Network Satisfaction Problem (NSP) for $\mathcal{P}$

Instance: a network $\mathcal{N} = (V; L)$, where $L : V^2 \to \mathcal{R}_2(\mathbb{Q})$.
Question: is there $f : V \to \mathbb{Q}$ such that for all (v_1, v_2) we have $(f(v_1), f(v_2)) \in L(v_1, v_2)$?

Network Satisfaction Problem for Point Algebra

A point algebra $\mathcal{P}$ is a set of all binary relations $\mathcal{R}_2(\mathbb{Q})$ with a first order definition over $(\mathbb{Q}; <)$:

- $<, >, \neq, \leq, \geq, =, \mathbb{Q}^2, \emptyset$.

The Network Satisfaction Problem (NSP) for $\mathcal{P}$

Instance: a network $\mathcal{N} = (V; L)$, where $L : V^2 \to \mathcal{R}_2(\mathbb{Q})$.
Question: is there $f : V \to \mathbb{Q}$ such that for all (v_1, v_2) we have $(f(v_1), f(v_2)) \in L(v_1, v_2)$?

Every instance of NSP($\mathcal{P}$) may be rewritten into an instance of CSP$(\mathbb{Q}; <, >, \neq, \leq, \geq, =, \mathbb{Q}^2, \emptyset)$.

Network Satisfaction Problem for Point Algebra

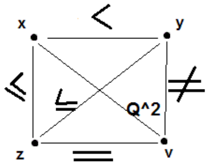
A point algebra $\mathcal{P}$ is a set of all binary relations $\mathcal{R}_2(\mathbb{Q})$ with a first order definition over $(\mathbb{Q}; <)$:

- $<, >, \neq, \leq, \geq, =, \mathbb{Q}^2, \emptyset$.

The Network Satisfaction Problem (NSP) for $\mathcal{P}$

Instance: a network $\mathcal{N} = (V; L)$, where $L : V^2 \to \mathcal{R}_2(\mathbb{Q})$.
Question: is there $f : V \to \mathbb{Q}$ such that for all (v_1, v_2) we have $(f(v_1), f(v_2)) \in L(v_1, v_2)$?

Every instance of NSP($\mathcal{P}$) may be rewritten into an instance of CSP$(\mathbb{Q}; <, >, \neq, \leq, \geq, =, \mathbb{Q}^2, \emptyset)$.



$$(x < y) \wedge (y \neq v) \wedge (z = v) \wedge (x \leq z) \wedge (z \leq y) \wedge (\mathbb{Q}^2(x, v))$$

is satisfiable in

$$(\mathbb{Q}; <, \leq, >, \geq, =, \neq, \emptyset, \mathbb{Q}^2).$$

is satisfiable iff

Consistency, Strong Consistency, and Global Consistency

Definition

An instance ϕ of CSP(Γ), for some Γ, over a set of variables V is:

- k-consistent if every partial solution to $(k-1)$ variables may be extended to any other variable;
- strongly k-consistent if it is i-consistent for every $i \leq k$;
- globally consistent if it is strongly $|V|$-consistent.

Consistency, Strong Consistency, and Global Consistency

Definition

An instance ϕ of CSP(Γ), for some Γ, over a set of variables V is:

- k-consistent if every partial solution to $(k-1)$ variables may be extended to any other variable;
- strongly k-consistent if it is i-consistent for every $i \leq k$;
- globally consistent if it is strongly $|V|$-consistent.

Example: (Koubarakis'95)

- $(x_1 \leq x_2) \wedge (x_2 \leq x_3) \wedge (x_4 \leq x_5) \wedge (x_5 \leq x_6) \wedge (x_2 \neq x_5)$ — 2-consistent but not 3-consistent.
  A partial solution: $x_1 = 3; x_3 = 0$ cannot be extended to x_2.

Consistency, Strong Consistency, and Global Consistency

Definition

An instance ϕ of CSP(Γ), for some Γ, over a set of variables V is:

- *k*-consistent if every partial solution to $(k-1)$ variables may be extended to any other variable;
- strongly *k*-consistent if it is i-consistent for every $i \leq k$;
- globally consistent if it is strongly $|V|$-consistent.

Example: (Koubarakis'95)

- $(x_1 \leq x_2) \wedge (x_2 \leq x_3) \wedge (x_4 \leq x_5) \wedge (x_5 \leq x_6) \wedge (x_2 \neq x_5)$ — 2-consistent but not 3-consistent.
  A partial solution: $x_1 = 3; x_3 = 0$ cannot be extended to x_2.
- But can be made (strongly) 3-consistent by adding $(x_1 \leq x_3) \wedge (x_4 \leq x_6)$.
  Now, $x_1 = 3; x_3 = 0$ is not a partial solution anymore.

Consistency, Strong Consistency, and Global Consistency

Definition

An instance ϕ of CSP(Γ), for some Γ, over a set of variables V is:

- k-consistent if every partial solution to $(k-1)$ variables may be extended to any other variable;
- strongly k-consistent if it is i-consistent for every $i \leq k$;
- globally consistent if it is strongly $|V|$-consistent.

Example: (Koubarakis'95)

- $(x_1 \leq x_2) \wedge (x_2 \leq x_3) \wedge (x_4 \leq x_5) \wedge (x_5 \leq x_6) \wedge (x_2 \neq x_5)$ — 2-consistent but not 3-consistent.
  A partial solution: $x_1 = 3; x_3 = 0$ cannot be extended to x_2.
- But can be made (strongly) 3-consistent by adding $(x_1 \leq x_3) \wedge (x_4 \leq x_6)$. Now, $x_1 = 3; x_3 = 0$ is not a partial solution anymore.
- Not 4-consistent. A part solution: $x_2 = x_4 = x_6 = 1$ can't be extended to x_5.

Consistency, Strong Consistency, and Global Consistency

Definition

An instance ϕ of CSP(Γ), for some Γ, over a set of variables V is:

- k-consistent if every partial solution to $(k-1)$ variables may be extended to any other variable;
- strongly k-consistent if it is i-consistent for every $i \leq k$;
- globally consistent if it is strongly $|V|$-consistent.

Example: (Koubarakis'95)

- $(x_1 \leq x_2) \wedge (x_2 \leq x_3) \wedge (x_4 \leq x_5) \wedge (x_5 \leq x_6) \wedge (x_2 \neq x_5)$ — 2-consistent but not 3-consistent.
  A partial solution: $x_1 = 3; x_3 = 0$ cannot be extended to x_2.
- But can be made (strongly) 3-consistent by adding $(x_1 \leq x_3) \wedge (x_4 \leq x_6)$.
  Now, $x_1 = 3; x_3 = 0$ is not a partial solution anymore.
- Not 4-consistent. A part solution: $x_2 = x_4 = x_6 = 1$ can't be extended to x_5.
- Can be made globally consistent by adding
  $(x_1 \neq x_5 \vee x_3 \neq x_5) \wedge (x_4 \neq x_2 \vee x_6 \neq x_2) \wedge (x_1 \neq x_3 \vee x_1 \neq x_4 \vee x_1 \neq x_6)$.

Establishing Strong Consistency

Theorem. (Bodirsky+Dalmau'06)

For every ω-categorical structure Γ and k there is an expansion Δ of Γ containing an empty relation R_e, and a polynomial-time algorithm which converts every instance ϕ of CSP(Γ) into an equivalent strongly k-consistent instance ψ of CSP(Δ). The algorithm tries to infer R_e.

Establishing Strong Consistency

Theorem. (Bodirsky+Dalmau'06)

For every ω-categorical structure Γ and k there is an expansion Δ of Γ containing an empty relation R_e, and a polynomial-time algorithm which converts every instance ϕ of CSP(Γ) into an equivalent strongly k-consistent instance ψ of CSP(Δ). The algorithm tries to infer R_e.

1. If ψ contains an occurrence of R_e, then ϕ does not have a solution.
2. For some Γ there exists k such that establishing strong k-consistency solves CSP(Γ), that is, the converse of 1 is also true.
3. For some Γ there exists k such that ψ is also globally consistent.

Establishing Strong Consistency

Theorem. (Bodirsky+Dalmau'06)

For every ω-categorical structure Γ and k there is an expansion Δ of Γ containing an empty relation R_e, and a polynomial-time algorithm which converts every instance ϕ of CSP(Γ) into an equivalent strongly k-consistent instance ψ of CSP(Δ). The algorithm tries to infer R_e.

1. If ψ contains an occurrence of R_e, then ϕ does not have a solution.
2. For some Γ there exists k such that establishing strong k-consistency solves CSP(Γ), that is, the converse of 1 is also true.
3. For some Γ there exists k such that ψ is also globally consistent.

Examples:

- $\Gamma = (\mathbb{Q}; \{(x, y, z) \mid x > y \vee x > z\})$ satisfies (1) but not (2). (Bodirsky+Kára'08)
- $\Gamma = (\mathbb{Q}; \neq, \{(x, y, z) \mid x \neq y \vee x = z\},)$ satisfies (2) but not (3). (e.g., this paper)
- $\Gamma = (\mathbb{Q}; <, \leq, \neq, \{(x_1, x_2, x_3, x_4) \mid x_1 \neq x_2 \vee x_3 \neq x_4\})$ satisfies (3) (Koubarakis'95)

Local-to-Global Consistency

Definition

We say that a structure Γ has local-to-global consistency w.r.t to k if every strongly k-consistent variant ψ of ϕ is also globally consistent.

Local-to-Global Consistency

Definition

We say that a structure Γ has local-to-global consistency w.r.t to k if every strongly k-consistent variant ψ of ϕ is also globally consistent.

Why local-to-global consistency is a desirable property?

Local-to-Global Consistency

Definition

We say that a structure Γ has local-to-global consistency w.r.t to k if every strongly k-consistent variant ψ of ϕ is also globally consistent.

Why local-to-global consistency is a desirable property?

- A set of constraints is globally consistent means:
  - it is completely explicit,
  - a solution may be found by greedily assigning values to the variables one at a time (without backtracking).

Local-to-Global Consistency

Definition

We say that a structure Γ has local-to-global consistency w.r.t to k if every strongly k-consistent variant ψ of ϕ is also globally consistent.

Why local-to-global consistency is a desirable property?

- A set of constraints is globally consistent means:
  - it is completely explicit,
  - a solution may be found by greedily assigning values to the variables one at a time (without backtracking).
- If Γ has local-to-global consistency, then global consistency may be achieved in polynomial time. In general, it may take exponential time.

Local-to-Global Consistency

Definition

We say that a structure Γ has local-to-global consistency w.r.t to k if every strongly k-consistent variant ψ of ϕ is also globally consistent.

Why local-to-global consistency is a desirable property?

- A set of constraints is globally consistent means:
  - it is completely explicit,
  - a solution may be found by greedily assigning values to the variables one at a time (without backtracking).
- If Γ has local-to-global consistency, then global consistency may be achieved in polynomial time. In general, it may take exponential time.

This property was studied in temporal reasoning by:

- Koubarakis'95: point algebra + disjunction of disequalities has local-to-global consistency, e.g. $(\mathbb{Q}; <, \leq, \neq, \{(x_1, x_2, x_3, x_4) \mid x_1 \neq x_2 \vee x_3 \neq x_4\})$.

Local-to-Global Consistency

Definition

We say that a structure Γ has local-to-global consistency w.r.t to k if every strongly k-consistent variant ψ of ϕ is also globally consistent.

Why local-to-global consistency is a desirable property?

- A set of constraints is globally consistent means:
  - it is completely explicit,
  - a solution may be found by greedily assigning values to the variables one at a time (without backtracking).
- If Γ has local-to-global consistency, then global consistency may be achieved in polynomial time. In general, it may take exponential time.

This property was studied in temporal reasoning by:

- Koubarakis'95: point algebra + disjunction of disequalities has local-to-global consistency, e.g. $(\mathbb{Q}; <, \leq, \neq, \{(x_1, x_2, x_3, x_4) \mid x_1 \neq x_2 \vee x_3 \neq x_4\})$.
- van Beek+Cohen'90, and Bessiére+Isli+Ligozat'96 identify subclasses of Allen's interval algebra for which establishing path-consistency implies global consistency.

ORD-Horn Languages

Definition

A structure $\Gamma = (\mathbb{Q}; R_1, \ldots, R_n)$ is called an ORD-Horn language if each R_i may be defined as a conjunction of ORD-Horn clauses each of which is of the form:

$$(x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee xRy), \text{ where } R \in \{<, \leq, =, \neq\}.$$

ORD-Horn Languages

Definition

A structure $\Gamma = (\mathbb{Q}; R_1, \ldots, R_n)$ is called an ORD-Horn language if each R_i may be defined as a conjunction of ORD-Horn clauses each of which is of the form:

$$(x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee xRy), \text{ where } R \in \{<, \leq, =, \neq\}.$$

ORD-Horn languages were heavily studied.

- By Bürckert and Nebel (1994) in the context of Allen's interval algebra.
- By Balbiani, Condotta and del Cerro (1998,1999,2002) in the context of the block algebras.
- Also by Ligozat (1994,1996); and many others.

ORD-Horn Languages

Definition

A structure $\Gamma = (\mathbb{Q}; R_1, \ldots, R_n)$ is called an ORD-Horn language if each R_i may be defined as a conjunction of ORD-Horn clauses each of which is of the form:

$$(x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee xRy), \text{ where } R \in \{<, \leq, =, \neq\}.$$

ORD-Horn languages were heavily studied.

- By Bürckert and Nebel (1994) in the context of Allen's interval algebra.
- By Balbiani, Condotta and del Cerro (1998,1999,2002) in the context of the block algebras.
- Also by Ligozat (1994,1996); and many others.

Local-to-Global Consistency in ORD-Horn

- For every ORD-Horn Language Γ there exists k such that establishing strong k-consistency solves CSP(Γ).
- Some ORD-Horn languages have local-to-global consistency, e.g., CSP($\mathbb{Q}; \leq, \{(x_1, x_2, x_3, x_4) \mid x_1 \neq x_2 \vee x_3 \neq x_4\}$) (Koubarakis'95).
- Some do not, e.g., CSP($\mathbb{Q}; \{(x_1, x_2, x_3) \mid x_1 \neq x_2 \vee x_1 = x_3\}$).

Basic ORD-Horn Languages and The Main Result of The Paper

Definition

An ORD-Horn language $\Gamma = (\mathbb{Q}; R_1, \ldots, R_k)$ is called Basic ORD-Horn if it can be defined as a conjunction of Basic ORD-Horn clauses each of which is in one of the following forms:

- (xRy), where $R \in \{<, \leq, =\}$,
- $(x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k)$, or
- $((x \neq x_1 \vee \cdots \vee x \neq x_k) \vee (x < y) \vee (y \neq y_1 \vee \cdots \vee y \neq y_l))$.

Basic ORD-Horn Languages and The Main Result of The Paper

Definition

An ORD-Horn language $\Gamma = (\mathbb{Q}; R_1, \ldots, R_k)$ is called Basic ORD-Horn if it can be defined as a conjunction of Basic ORD-Horn clauses each of which is in one of the following forms:

- (xRy), where $R \in \{<, \leq, =\}$,
- $(x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k)$, or
- $((x \neq x_1 \vee \cdots \vee x \neq x_k) \vee (x < y) \vee (y \neq y_1 \vee \cdots \vee y \neq y_l))$.

The main result of this paper.

Theorem

Let Γ be an ORD-Horn language. Then the following are equivalent.

1. Γ *has local-to-global consistency.*
2. Γ *is Basic ORD-Horn.*

In the Proof of the Main Theorem We Use: Polymorphisms, ...

Definition of a polymorphism

Let $R \subseteq \mathbb{Q}^n$. A function $f : \mathbb{Q}^m \to \mathbb{Q}$ is a polymorphism of R if:

for all tuples $a^1, \ldots, a^m \in R$ of the form

$$\begin{bmatrix} (a_1^1, & \ldots & , a_n^1) & \in R \\ \vdots & & \vdots & \\ (a_1^m & \ldots & , a_n^m) & \in R \end{bmatrix}$$

it holds that $(f(a_1^1, \ldots, a_1^m), \ldots, f(a_n^1, \ldots, a_n^m)) \in R$.

A function f is a polymorphism of Γ if it is a polymorphism of every R in Γ.

In the Proof of the Main Theorem We Use: Polymorphisms, ...

Definition of a polymorphism

Let $R \subseteq \mathbb{Q}^n$. A function $f : \mathbb{Q}^m \to \mathbb{Q}$ is a polymorphism of R if:

for all tuples $a^1, \ldots, a^m \in R$ of the form $\begin{bmatrix} (a_1^1, & \ldots & , a_n^1) & \in R \\ \vdots & & \vdots & \\ (a_1^m & \ldots & , a_n^m) & \in R \end{bmatrix}$

it holds that $(f(a_1^1, \ldots, a_1^m), \ldots, f(a_n^1, \ldots, a_n^m)) \in R$.

A function f is a polymorphism of Γ if it is a polymorphism of every R in Γ.

Definition

A k-ary function $f : \mathbb{Q}^k \to \mathbb{Q}$ where $k \geq 3$ is called a quasi near-unanimity function (QNUF) if and only if it satisfies

$$\forall x \forall y.\ f(y, x, x, \ldots, x) = f(x, y, x, \ldots, x) = \cdots = f(x, x, x, \ldots, y) = f(x, \ldots, x)$$

In the Proof of the Main Theorem We Use: Polymorphisms, ...

Definition of a polymorphism

Let $R \subseteq \mathbb{Q}^n$. A function $f : \mathbb{Q}^m \to \mathbb{Q}$ is a polymorphism of R if:

for all tuples $a^1, \ldots, a^m \in R$ of the form $\begin{bmatrix} (a_1^1, & \ldots & , a_n^1) & \in R \\ \vdots & & \vdots & \\ (a_1^m & \ldots & , a_n^m) & \in R \end{bmatrix}$

it holds that $(f(a_1^1, \ldots, a_1^m), \ldots, f(a_n^1, \ldots, a_n^m)) \in R$.

A function f is a polymorphism of Γ if it is a polymorphism of every R in Γ.

Definition

A k-ary function $f : \mathbb{Q}^k \to \mathbb{Q}$ where $k \geq 3$ is called a quasi near-unanimity function (QNUF) if and only if it satisfies

$$\forall x \forall y.\, f(y, x, x, \ldots, x) = f(x, y, x, \ldots, x) = \cdots = f(x, x, x, \ldots, y) = f(x, \ldots, x)$$

Theorem. (Bodirsky+Dalmau'06)

Let $k \geq 3$. An ω-categorical structure Γ has a k-ary oligopotent QNUF as a polymorphism if and only if Γ has local-to-global consistency w.r.t. k.

and decomposability.

Definition

An n-ary relation R is k-decomposable if it contains all tuples t such that for every subset I of $\{1, \ldots, n\}$ with $|I| \leq k$ there is a tuple $s \in R$ such that $t[i] = s[i]$ for all $i \in I$. A structure Γ is k-decomposable if every relation in Γ is k-decomposable.

and decomposability.

Definition

An n-ary relation R is k-decomposable if it contains all tuples t such that for every subset I of $\{1, \ldots, n\}$ with $|I| \leq k$ there is a tuple $s \in R$ such that $t[i] = s[i]$ for all $i \in I$. A structure Γ is k-decomposable if every relation in Γ is k-decomposable.

Example: $R = \{(0,1,0,0),(0,0,1,0),(1,0,0,0)\}$ is not 2-decomposable. It does not contain $(0,0,0,0)$.

and decomposability.

Definition

An n-ary relation R is k-decomposable if it contains all tuples t such that for every subset I of $\{1, \ldots, n\}$ with $|I| \leq k$ there is a tuple $s \in R$ such that $t[i] = s[i]$ for all $i \in I$. A structure Γ is k-decomposable if every relation in Γ is k-decomposable.

Example: $R = \{(0,1,0,0),(0,0,1,0),(1,0,0,0)\}$ is not 2-decomposable. It does not contain $(0,0,0,0)$.

Theorem. (Bodirsky+Chen'07)

An ω-categorical structure Γ has local-to-global consistency w.r.t. k if and only if Γ is $(k-1)$-decomposable.

and decomposability.

Definition

An n-ary relation R is k-decomposable if it contains all tuples t such that for every subset I of $\{1, \ldots, n\}$ with $|I| \leq k$ there is a tuple $s \in R$ such that $t[i] = s[i]$ for all $i \in I$. A structure Γ is k-decomposable if every relation in Γ is k-decomposable.

Example: $R = \{(0,1,0,0),(0,0,1,0),(1,0,0,0)\}$ is not 2-decomposable. It does not contain $(0,0,0,0)$.

Theorem. (Bodirsky+Chen'07)

An ω-categorical structure Γ has local-to-global consistency w.r.t. k if and only if Γ is $(k-1)$-decomposable.

To obtain the result we show that:

- every Basic ORD-Horn language has an oligopotent QNUF as a polymorphism, and that
- every other ORD-Horn language is not k-decomposable for any k.

The End

Thank you for your attention.