Breaking variable symmetry in almost injective problems

Philippe Vismara[1,2] Rémi Coletta[1]

[1] LIRMM - [2] Montpellier SupAgro, MISTEA - France

18th International Conference on Principles and Practice of Constraint Programming, Québec, 2012

Motivation

How many Lex-leader constraints to Break Variable Symmetries ?

- one constraint for each symmetry
  $\rightsquigarrow$ exponential number of constraints
- for injective problems (Alldiff)
  $\rightsquigarrow$ linear number of constraints [Puget 2005]

What about "almost injective" problems ?
Can we determine the number of constraints in the framework of parametrized complexity ?

Trivial result :
the number of Lex constraints is FPT in the number of symmetries

Almost injective problems

- injective : `AllDiff` constraint
- almost injective problem : `GCC` constraint

A parameter to characterize almost injective problems

μ = maximum number of variables that can be equal simultaneously

- injective problem : $\mu = 0$
- almost injective problem : μ small

Almost injective problems

- injective : `AllDiff` constraint
- almost injective problem : `GCC` constraint

A parameter to characterize almost injective problems

μ = maximum number of variables that can be equal simultaneously

- injective problem : $\mu = 0$
- almost injective problem : μ small

GCC (Global Cardinality Constraint)

number of variables assigned to value v is between $lb(v)$ and $ub(v)$.

$$\mu \leq \sum_{v \in D \text{ s.t. } ub(v) > 1} ub(v)$$

Lex-leader constraints on almost injective problems

Lex constraint for symmetry σ :

$X_{i_1}, X_{i_2}, .., \quad \leq_{lex} X_{\sigma(i_1)}, X_{\sigma(i_2)}, ..,$

Lex-leader constraints on almost injective problems

Lex constraint for symmetry σ :

$x_{i_1}, x_{i_2}, .., \leq_{lex} x_{\sigma(i_1)}, x_{\sigma(i_2)}, ..,$

- injective problem: $x_i = x_{\sigma(i)} \Rightarrow \sigma(i) = i$

Lex-leader constraints on almost injective problems

Lex constraint for symmetry σ :

$x_{i_1}, x_{i_2}, .., \leq_{lex} x_{i_1}, x_{\sigma(i_2)}, ..,$

- injective problem: $x_i = x_{\sigma(i)} \Rightarrow \sigma(i) = i$

Lex-leader constraints on almost injective problems

Lex constraint for symmetry σ :

$x_{i_1}, x_{i_2}, .., \leq_{lex} x_{i_1}, x_{i_2}, ..,$

- injective problem: $x_i = x_{\sigma(i)} \Rightarrow \sigma(i) = i$

Lex-leader constraints on almost injective problems

Lex constraint for symmetry σ :

$$x_{i_1}, x_{i_2}, .., x_{i_\sigma^1} \leq_{lex} x_{i_1}, x_{i_2}, .., x_{\sigma(i_\sigma^1)}$$

- injective problem: $x_i = x_{\sigma(i)} \Rightarrow \sigma(i) = i$

Lex-leader constraints on almost injective problems

Lex constraint for symmetry σ :

$x_{i_1}, x_{i_2}, .., x_{i^1_\sigma} \leq_{lex} x_{i_1}, x_{i_2}, .., x_{\sigma(i^1_\sigma)}$

- injective problem: $x_i = x_{\sigma(i)} \Rightarrow \sigma(i) = i$
- $x_{i^1_\sigma} \leq x_{\sigma(i^1_\sigma)}$
  where i^1_σ = first index such that $i^1_\sigma \neq \sigma(i^1_\sigma)$

Lex-leader constraints on almost injective problems

Lex constraint for symmetry σ :

$$x_{i_1}, x_{i_2}, .., x_{i^1_\sigma}, .., x_{i^2_\sigma} \quad \leq_{lex} \quad x_{i_1}, x_{i_2}, .., x_{\sigma(i^1_\sigma)}, .., x_{\sigma(i^2_\sigma)}$$

- injective problem: $x_i = x_{\sigma(i)} \Rightarrow \sigma(i) = i$
- $x_{i^1_\sigma} \leq x_{\sigma(i^1_\sigma)}$
  where i^1_σ = first index such that $i^1_\sigma \neq \sigma(i^1_\sigma)$
- one duplicate value:
  $x_{i^1_\sigma} = x_{\sigma(i^1_\sigma)} \rightarrow x_{i^2_\sigma} \leq x_{\sigma(i^2_\sigma)}$
  where $i^2_\sigma = 2^d$ index such that $i^2_\sigma \neq \sigma(i^2_\sigma)$

Lex-leader constraints on almost injective problems

Lex constraint for symmetry σ :

$$x_{i_1}, x_{i_2}, .., x_{i^1_\sigma}, .., x_{i^2_\sigma}, .., x_{i^\mu_\sigma} \leq_{lex} x_{i_1}, x_{i_2}, .., x_{\sigma(i^1_\sigma)}, .., x_{\sigma(i^2_\sigma)}, .., x_{\sigma(i^\mu_\sigma)}$$

- injective problem: $x_i = x_{\sigma(i)} \Rightarrow \sigma(i) = i$
- $x_{i^1_\sigma} \leq x_{\sigma(i^1_\sigma)}$
  where i^1_σ = first index such that $i^1_\sigma \neq \sigma(i^1_\sigma)$
- one duplicate value:
  $x_{i^1_\sigma} = x_{\sigma(i^1_\sigma)} \rightarrow x_{i^2_\sigma} \leq x_{\sigma(i^2_\sigma)}$
  where i^2_σ = 2^d index such that $i^2_\sigma \neq \sigma(i^2_\sigma)$
- μ simultaneous pairs of equal variables:
  $x_{i^1_\sigma} = x_{\sigma(i^1_\sigma)} \wedge \ldots \wedge x_{i^{\mu-1}_\sigma} = x_{\sigma(i^{\mu-1}_\sigma)} \rightarrow x_{i^\mu_\sigma} \leq x_{\sigma(i^\mu_\sigma)}$

Almost injective problems

With no more than μ simultaneous pairs of equal variables Lex constraint simplifies to :

$$x_{i_\sigma^1}, x_{i_\sigma^2}, \ldots, x_{i_\sigma^\mu} \leq_{lex} x_{\sigma(i_\sigma^1)}, x_{\sigma(i_\sigma^2)}, \ldots, x_{\sigma(i_\sigma^\mu)} \quad (1)$$

where $i_\sigma^1, i_\sigma^2, \ldots i_\sigma^\mu$ are the μ indexes such that $\sigma(i) \neq i$.

Almost injective problems

With no more than μ simultaneous pairs of equal variables Lex constraint simplifies to :

$$x_{i_\sigma^1}, x_{i_\sigma^2}, \ldots, x_{i_\sigma^\mu} \leq_{lex} x_{\sigma(i_\sigma^1)}, x_{\sigma(i_\sigma^2)}, \ldots, x_{\sigma(i_\sigma^\mu)} \quad (1)$$

where $i_\sigma^1, i_\sigma^2, \ldots i_\sigma^\mu$ are the μ indexes such that $\sigma(i) \neq i$.

$\leadsto$ number of constraints in $\mathcal{O}(\binom{n}{\mu}) < \mathcal{O}(n^\mu)$ hence **XP in** μ

Almost injective problems

If no more than ν variables can take duplicate values:

- reorder variables to put them at the end

$\rightsquigarrow$ number of constraints in $\mathcal{O}(\nu^{\mu} + n)$ hence **FPT in μ and ν**

Heterogeneous domains

Example

if $D(x_1) \cap D(x_2) = \{v\} = D(x_3) \cap D(x_4)$ and $ub(v) = 3$, then `Gcc` forbids $x_1 = x_2$ and $x_3 = x_4$ simultaneously.

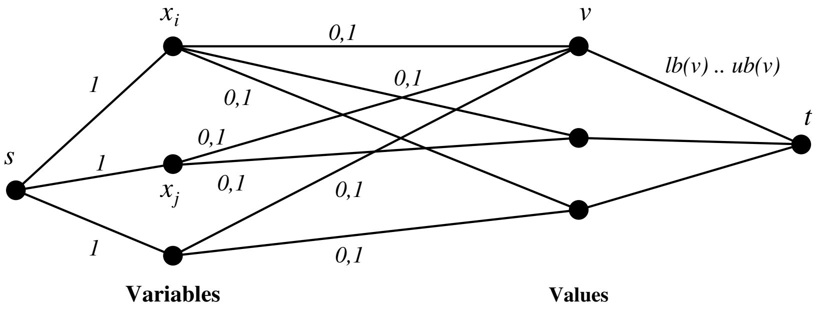
- Some constraints can be discarded with the GCC

Heterogeneous domains

- Before posting constraint
  $(x_{i_\sigma^1} = x_{\sigma(i_\sigma^1)}) \wedge \ldots \wedge (x_{i_\sigma^{\rho-1}} = x_{\sigma(i_\sigma^{\rho-1})}) \rightarrow x_{i_\sigma^\rho} \leq x_{\sigma(i_\sigma^\rho)}$
- Solve subproblem $N' =< X, D, C' >$ with
  $C' = \{x_{i_\sigma^1} = x_{\sigma(i_\sigma^1)}, \ldots, x_{i_\sigma^{\rho-1}} = x_{\sigma(i_\sigma^{\rho-1})}\} \cup \{Gcc\}$
- Test based on flow:

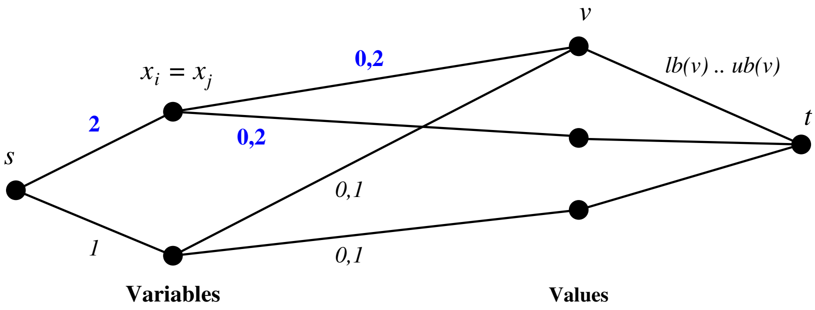
Heterogeneous domains

- Before posting constraint
  $(x_{i_\sigma^1} = x_{\sigma(i_\sigma^1)}) \wedge \ldots \wedge (x_{i_\sigma^{\rho-1}} = x_{\sigma(i_\sigma^{\rho-1})}) \rightarrow x_{i_\sigma^\rho} \leq x_{\sigma(i_\sigma^\rho)}$
- Solve subproblem $N' =< X, D, C' >$ with
  $C' = \{x_{i_\sigma^1} = x_{\sigma(i_\sigma^1)}, \ldots, x_{i_\sigma^{\rho-1}} = x_{\sigma(i_\sigma^{\rho-1})}\} \cup \{Gcc\}$
- Test based on flow:



Heterogeneous domains

- Before posting constraint $(x_{i_\sigma^1} = x_{\sigma(i_\sigma^1)}) \wedge \ldots \wedge (x_{i_\sigma^{\rho-1}} = x_{\sigma(i_\sigma^{\rho-1})}) \rightarrow x_{i_\sigma^\rho} \leq x_{\sigma(i_\sigma^\rho)}$
- Solve subproblem $N' =< X, D, C' >$ with $C' = \{x_{i_\sigma^1} = x_{\sigma(i_\sigma^1)}, \ldots, x_{i_\sigma^{\rho-1}} = x_{\sigma(i_\sigma^{\rho-1})}\} \cup \{Gcc\}$
- Test based on flow:



Conclusion

Number of Lex constraints to break variable symmetries

- XP in μ, where μ is the number of variables simultaneously equal
- FPT in μ and ν when at most ν variables are assigned to duplicated values
- test to discard useless constraints

Perspectives :

- identify new parameters
- apply on graph problems