Containment, Equivalence and Coreness from CSP to QCSP and beyond

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Model-checking

We are interested in the parameterisation of the model checking problem by the model. Fix a logic $\mathscr{L}$ and fix $\mathcal{D}$.

The problem "$\mathscr{L}(\mathcal{D})$" has

- Input: a sentence φ of $\mathscr{L}$.
- Question: does $\mathcal{D} \models \varphi$?

We consider syntactic fragments $\mathscr{L}$ of FO.

- For $\mathscr{L} = \{\exists, \wedge, =\}$ this is the Constraint Satisfaction Problem (CSP).
- For $\mathscr{L} = \{\forall, \exists, \wedge, =\}$ this is the Quantified CSP (QCSP).
- For $\mathscr{L} = \{\forall, \exists, \wedge, \vee\}$ this is some other strange problem I studied.

What is Core-ness?

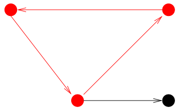
(?) Call a structure $\mathcal{D}$ an $\mathscr{L}$-core if it is minimal w.r.t. size among structures that agree on $\mathscr{L}$.

(?) Call a structure $\mathcal{D}$ an $\mathscr{L}$-core if for no proper substructure $\mathscr{D}'$ do $\mathscr{D}'$ and $\mathscr{D}$ agree on $\mathscr{L}$.

For CSP, the $\{\exists, \wedge, =\}$-core is the *core*!

- Both definitions above coincide.

The *core of* $\mathcal{D}$ is a minimal induced substructure $\mathcal{X} \subseteq \mathcal{D}$ all of whose endomorphisms are automorphisms.



It is well-known that $\mathcal{X}$ is unique up to iso and $\mathrm{CSP}(\mathcal{D}) = \mathrm{CSP}(\mathcal{X})$.

The $\{\forall, \exists, \wedge, \vee\}$-core, the so-called U-X-core, is again well-behaved.

- The two definitions coincide. It is known to be unique up to iso and be a minimal induced substructure.

The $\{\forall, \exists, \neg, \wedge, \vee, =\}$-core is clearly well-behaved.

- Every structure is a $\{\forall, \exists, \neg, \wedge, \vee, =\}$-core!

In fact, the $\{\forall, \exists, \wedge, \vee, =\}$-core is equally well-behaved.

- Every structure is a $\{\forall, \exists, \wedge, \vee, =\}$-core!

The point of cores

In CSPs, restriction to cores enables one to assume

- constants naming the elements
- that the corresponding algebras are idempotent

What are the properties of $\{\forall, \exists, \wedge\}$-cores?

- For one thing, the two definitions do not coincide.

$\mathcal{A}$ $\mathcal{B}$

Both $\mathcal{A}$ and $\mathcal{B}$ are $\mathscr{L}$-cores! But only $\mathcal{B}$ is a $\mathscr{L}$-core.

We will revert to the second definition.

- Call a structure $\mathcal{D}$ a Q-core if for no proper substructure $\mathscr{D}'$ do $\mathscr{D}'$ and $\mathscr{D}$ agree on $\{\forall, \exists, \wedge, =\}$.

This also gives the natural notion for *Q-core of*.

Questions:

- Is this notion useful?
- Is the Q-core of a structure unique up to iso?

Answers

Q-cores are useful for simplifying classifications!

- If $\mathcal{H}$ is a partially reflexive forest, then either the Q-core of $\mathcal{H}$ has a majority polymorphism and QCSP($\mathcal{H}$) is in P, or QCSP($\mathcal{H}$) is NP-hard.

Uniqueness remains unknown. We conjecture the Q-core is unique up to iso.

- Can we reduce to the idempotent ???

<table>
<tr><th>$\{\exists, \forall, \wedge, \vee, =\}$-FO</th><th>$\{\exists, \forall, \wedge, \vee\}$-FO</th><th>$\{\exists, \forall, \wedge\}$-FO</th><th>$\{\exists, \wedge\}$-FO</th></tr>
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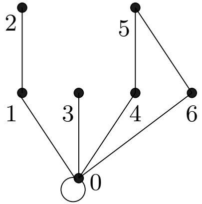
<td>$\mathcal{A}_4$
</td>
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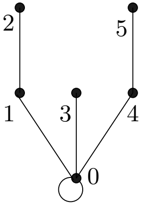</td>
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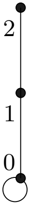</td>
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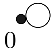</td>
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<tr><td>isomorphism</td><td>U-X-Core</td><td>Q-core</td><td>Core</td></tr>
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Table : different notions of "core" (the circles represent self-loops).