Solving Minimal Constraint Networks in Qualitative Spatial and Temporal Reasoning

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Minimal networks

- Complete networks in which *each* tuple in any constraint can be extended to a solution

x {(1,2), <u>(1,3)</u>} y, y {(2,1), <u>(3,2)</u>} z, x {(1,1)} z

Solution: $x = 1, y = 2, z = 1$

Not minimal: $x = 1, y = 3$ can not be extended

  – A minimal network is satisfiable

- Computing a solution of a minimal network is NP-hard [Gottlob CP11]
  – If the above problem is in P, then SAT is also in P
- ***How is the case in QSTR?***

Outline

- *Qualitative spatial and temporal reasoning*
  - Qualitative calculi
    - *Interval Algebra, Cardinal Relation Algebra, RCC-5/8*
  - CSP in QSTR
- *Main result*:

  Computing a solution of minimal networks in above qualitative calculi is NP-hard.
- *Proof sketch*

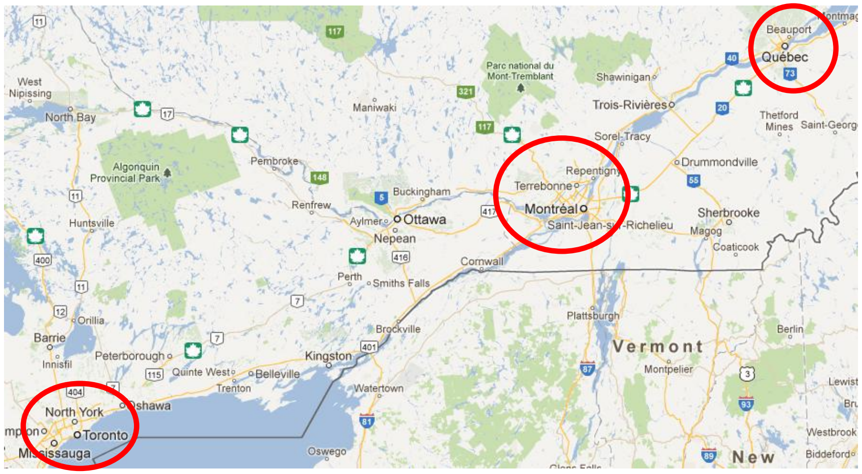
What's QSTR ?

- *Qualitative Spatial and Temporal Reasoning* :

  <u>Represent</u> and <u>reason with</u> spatial (or temporal) knowledge in a qualitative manner

  - Québec City is to the northeast of Montreal
  - Montreal is to the northeast of Toronto

- Vs. quantitative approach
  - High level
  - Closer to human cognition



Qualitative calculus

- Language in QSTR
- Dealing with a certain aspect (e.g., *topology*, *direction*, *size*) of space or time
- Characterized by its universe and basic relations

Universe : U

Set of spatial (or temporal) entities

Basic relations :

$$B = \{b_1, b_2, \ldots, b_n\}$$

A partition of U^k (in this work k=2)

Point Algebra

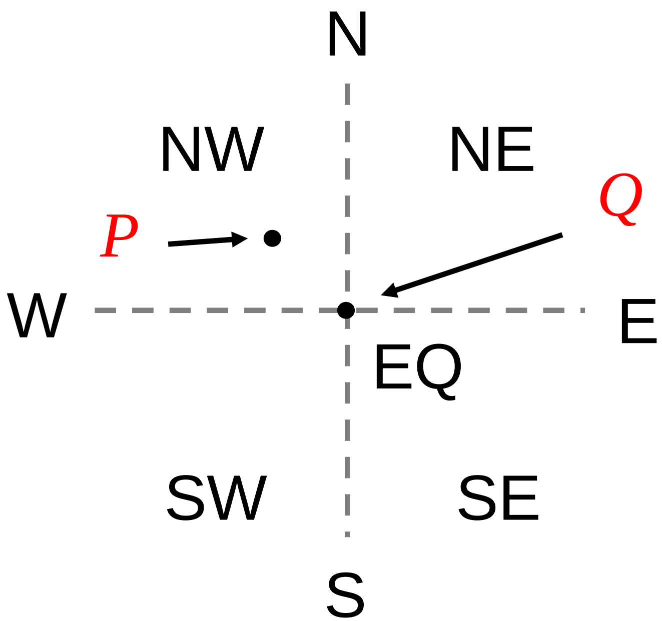
U : Real numbers

B : $\{<, =, >\}$

Cardinal Relation Algebra

- Universe : Real plane
- Basic relations:

NW, N, NE, W, EQ, E, SW, S, SE

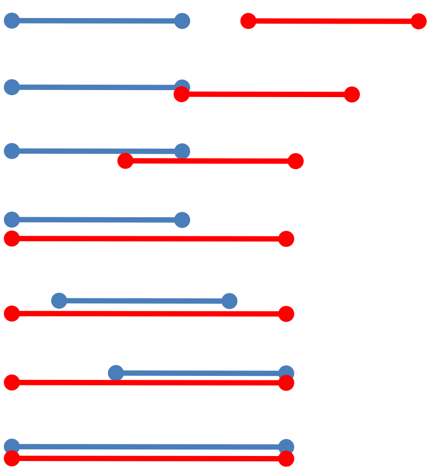


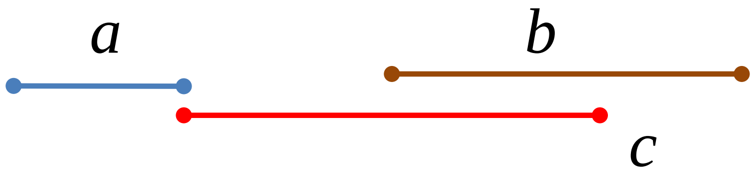
P NW Q

Interval Algebra

- Universe : all closed intervals $[x^-, x^+]$
- 13 basic relations

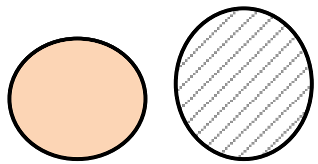
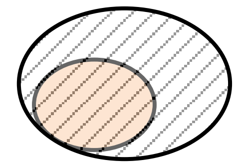
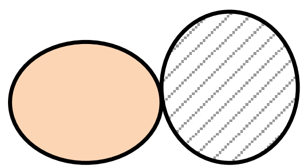
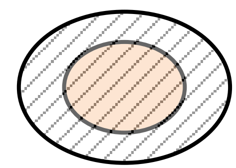
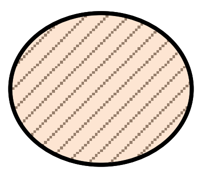
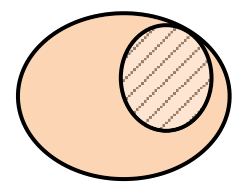
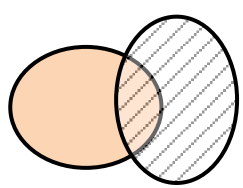
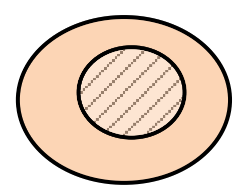
Relation	Symbol	Converse	Meaning
before	p	pi	$x^- < x^+ < y^- < y^+$
meets	m	mi	$x^- < x^+ = y^- < y^+$
overlaps	o	oi	$x^- < y^- < x^+ < y^+$
starts	s	si	$x^- = y^- < x^+ < y^+$
during	d	di	$y^- < x^- < x^+ < y^+$
finishes	f	fi	$y^- < x^- < x^+ = y^+$
equals	eq	eq	$x^- = y^- < x^+ = y^+$





RCC-8 and RCC-5

Universe: *plane regions*

DC (disconnected)		*TPP* (tangential proper part)	
EC (externally connected)		*NTPP* (non-tangential proper part)	
EQ (equal)		*TPPi* (tangential proper part inverse)	
PO (partially overlapping)		*NTPPi* (non-tangential proper part inverse)	

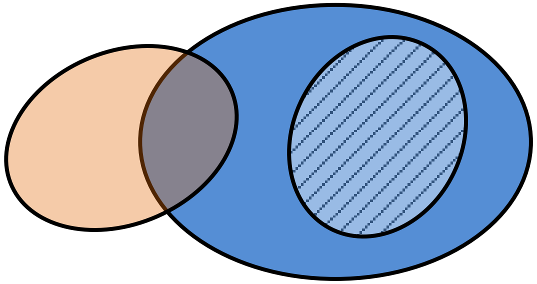
DR (DC, EC)

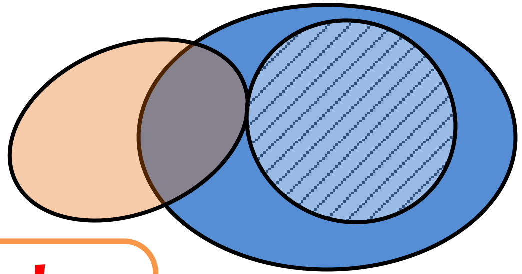
PP (TPP, NTPP)

PPi (TPPi, NTPPi)

CSP in QSTR

- Qualitative calculus: a constraint language
  - The domain of each variable is the universe
  - The relation in each constraint is a set of basic relations
- Examples:
  - x {DC, EC} y, y NTPP z, x PO z
    - Satisfiable, minimal
  - $v_1 < v_2,\ v_2\{<,=\}v_3,\ v_1\{<,>\}v_3$
    - Satisfiable, not minimal
    - $v_1 > v_3$ can not be satisfied





Complexity of solving minimal networks (in different qualitative calculi) in QSTR?

Minimal networks in QSTR

- For CRA, IA, RCC-5/8, the answer is NP-hard
- Though Gottlob's approach for proving the NP-hardness is followed, our result is not implied directly:

  Domains of variables

  free VS. assumed (infinite)

  Constraints

  free VS. restricted (basic relations as tuples)

Proof technique *(Gottlob 11)*

Solving a minimal network is NP-hard:

If there is a polynomial algorithm $\mathcal{A}$ that computes a solution of a minimal network, then some NP-hard problem $\mathcal{P}$ can be solved in polynomial time by an algorithm based on $\mathcal{A}$ …

Provided that there exists a polynomial mapping R from $\mathcal{P}$ to CSP, such that for instance φ in problem $\mathcal{P}$,

(*) φ is positive iff $R(\varphi)$ is a minimal network; and

(**) φ is negative iff $R(\varphi)$ is an unsatisfiable network.

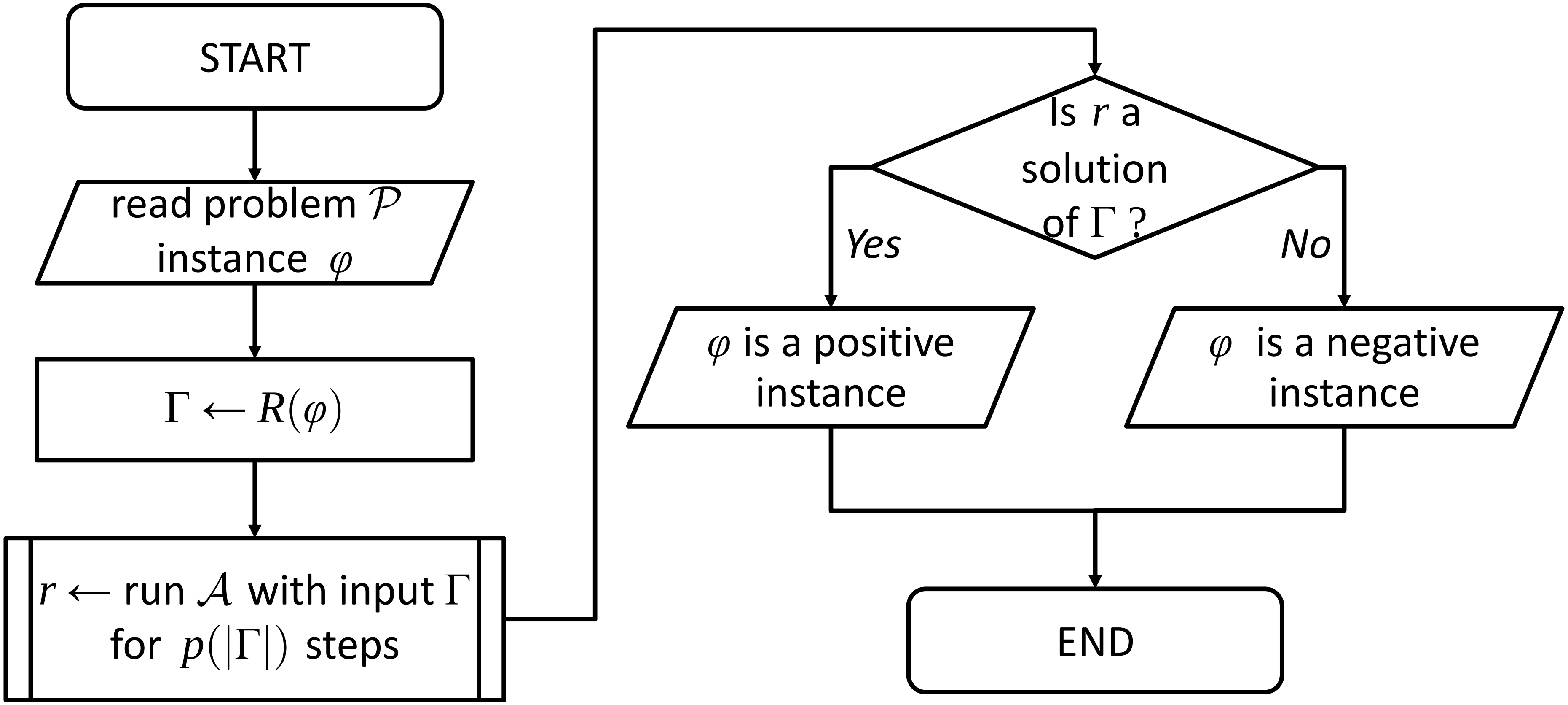
R is a reduction, therefore, deciding the minimality of CSP is NP-hard.

Proof technique (Gottlob 11)

(1) $\mathcal{A}$ computes a solution of a minimal network in $p(.)$ time

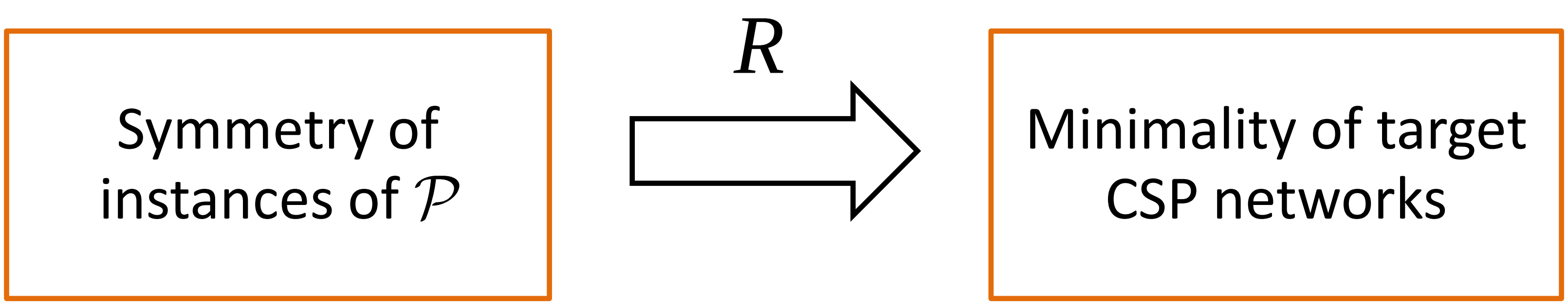
(2) φ is positive (negative) iff $R(\varphi)$ is minimal (unsatisfiable)

The following polynomial algorithm solves NP-hard problem $\mathcal{P}$



To prove the NP-hardness…

- Find an NP-hard problem $\mathcal{P}$, and
- A reduction R from $\mathcal{P}$ to the target CSP, s.t.
  - Positive instances mapped to minimal networks
  - Negative instances mapped to unsatisfiable networks.



Symmetric SAT

A SAT instance φ is symmetric if either φ is unsatisfiable, or for any an assignment π, π satisfies φ implies that assignment $\overline{\pi}$ is also satisfying, where π assigns each propositional variable p the opposite truth value to $\overline{\pi}(p)$.

Lemma

A SAT instance φ can be transformed in polynomial time into a symmetric SAT instance φ^*, preserving satisfiability.

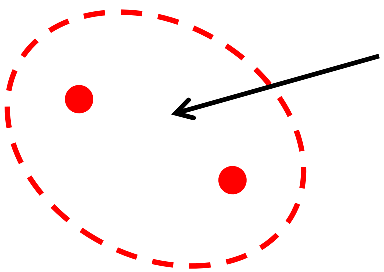
$$\varphi = (p_1 \vee \neg p_2) \wedge (p_2 \vee p_3)$$

$$\varphi^* = (p_1 \vee \neg p_2 \vee q) \wedge (p_2 \vee p_3 \vee q) \wedge (\neg p_1 \vee p_2 \vee \neg q) \wedge (\neg p_2 \vee \neg p_3 \vee \neg q)$$

Symmetric SAT is NP-hard

Proof sketch, Cardinal Relation Algebra

Propositional variable p_i $\Longrightarrow$ spatial variables x_i and y_i



x_i, y_i

x_i {NW, SE} y_j

p_i is assigned *true*

p_i is assigned *false*

Literal l $\Longrightarrow$ spatial variables c and d

l is assigned *true* (*false*) : c is to the left (right) of d

$$c <_x (>_x) d$$

Proof sketch, Cardinal Relation Algebra

- For clause c contains literals $l_1, \ldots, l_t$
  We have spatial variables $c_1, d_1, \ldots, c_t, d_t$
- We impose constraints such that c_{k+1}, d_k are on the same vertical line (c_{t+1} considered as c_1)

$$c_{k+1} =_x d_k$$

- All literals are assigned false :

$$c_1 >_x d_1 =_x c_2 >_x d_2 =_x \ldots =_x c_t >_x d_t =_x c_1$$

- Symmetry of the SAT instances also forbids the case that all literals are assigned true.
- The constructed CRA network is minimal if SAT instance is satisfiable.

Interval Algebra

- An interval $[x, y]$ corresponds to a point (x, y)
- Translate previous reduction

NW	N	NE	W	EQ	E	SW	S	SE
di	si	oi	fi	eq	f	o	s	d

RCC-5/8

k-supersymmetric **SAT** [Gottlob 11] **:**

A SAT instance φ is k-supersymmetric if either φ is unsatisfiable, or arbitrary partial truth value assignment over k variables can be extended to a satisfying assignment of φ.

Lemma

A SAT instance φ can be transformed in polynomial time into a symmetric and k-supersymmetric SAT instance φ^*, preserving satisfiability.

A more delicate reduction is needed… refer to the paper for details.

Conclusion

- *Solving a minimal network in qualitative calculi IA, CRA, RCC-5/8 is NP-hard.*
- *Bi-product: deciding minimality in these qualitative calculi is NP-hard.*

- Thank you for your attention !
- Questions?